

On partita doppia

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1 Introduction.

1.1 Double-entry bookkeeping

In 1494 Fra Luca Pacioli published in Venice one of the first printed mathematical books [P1494]. One section, *Computis e Scripturis*, is the first published description of *partita doppia* or double-entry bookkeeping, the foundation of accounting. Double-entry bookkeeping had been developed over a period of 200 years by Italian merchants and bankers. The aim of accounting is the measurement of a distributed concurrent system, and it is our contention that it is one of the earliest and most successful mathematical theories of concurrency. It is interesting that at that time negative numbers were not accepted; in the following century mathematicians repaired this deficiency though there seems to have been no attempt to explicate the distributed algebra underlying partita doppia. For further information about early accounting see [BJ84], [DeR56].

In this paper we give a precise mathematical account of partita doppia in terms of an algebraic structure on **Span(RGraph)** - the bicategory of spans of reflexive graphs. Some interesting new mathematical considerations arise in the study of this example. The fact that accounting concerns the measurement of

distributed systems is explicit in the form of the mathematics. We need to consider **Span(RGraph)** as a not-necessarily self-dual compact closed bicategory in order to account for the direction of flow of value. Another new aspect is that accounts give an example of a compositional concept represented by a lax functor: we are aware of many other compositional concepts similarly represented and believe that this is a widespread phenomenon, one which we will discuss in further detail in a complete version of this paper. There is also an interesting new mathematical result regarding the compact-closedness of a certain lax comma construction.

Another aspect to note in this work is the essential use of the two dimensional structure of **Span(RGraph)**. Locally ordered bicategories (whose arrows are ‘relations extended in time’) have been used by Abramsky to model interaction [Abr93], but in this paper a local order is insufficient: the crucial totalling of values in a system of accounts is a genuine 2-cell.

1.2 Pacioli

Pacioli is an interesting figure, born in Borgo San Sepolcro (now Sansepolcro) circa 1452, he was a student of another son of that city, Piero della Francesca. He taught in Florence, Rome, Perugia, Milan, Venice and Bologna. He was a friend of Leonardo, and in fact they collaborated on the book, *De Divina Proportione* (Venezia, 1509). His *Summa* was highly successful and assisted the dispersion of mathematics, contributing in this way to the flowering of mathematics in Europe in the 16th century. The book is a compilation of known results. Pacioli is known as the father of accounting.

Vasari in *The lives of the artists* (1568), was suspicious of Pacioli’s originality: From the chapter of Vasari [V1568] on Piero della Francesca:

“The Franciscan, Luca dal Borgo, who wrote about the regular geometrical bodies, was his [Piero della Francesca’s] pupil; and when Piero died at an advanced age, having written many books, this Luca arrogated them to himself and published as his own work what had fallen into his hands after his master’s death.”

One section of *De Divina Proportione* seems to be essentially due to Piero della Francesca.

1.3 Prerequisites

Graphs have been used since time immemorial to model systems, with the vertices representing *states* and the edges representing *atomic actions* of the system. Recently (in [KSW97b], [KSW97c]) the authors have proposed algebraic structures on **Span(RGraph)** as a way of modelling non-deterministic concurrent systems. We recall here briefly the ideas in this model but refer for more details to those two papers.

An initial version of this work on *partita doppia* was described by the third author at New Trends in Semantics, Bologna, 4-5 July, 1997.

This work arose out of earlier study of bicategories of processes [KSW97a], [K96] which itself arose from the study of distributive categories and imperative programming [W92a], [W92b], [KW93].

Much of the notation in this paper originates in category theory; for this the reader may consult a text such as [M70] or [W92a].

1.4 Acknowledgment

We are grateful in particular for conversations with Henry Weld. We acknowledge also the support for this project by the Australian Research Council, Italian MURST 40% and the Italian CNR.

2 Span(RGraph)

The algebra of this paper is the compact-closed symmetric-monoidal bicategory **Span(RGraph)**. Although these notions are well-known in the categorical literature (compact-closed [KL80], monoidal [M70], bicategory [B67], Span [B67], reflexive graphs [Law89]) we will describe them briefly here at the same time introducing a pictorial representation for expressions of arrows.

2.1 The algebra of graphs

The algebra **Span(RGraph)** consists of three sorts of things, objects, arrows and 2-cells. We will list the operations of the algebra as we describe its elements.

2.1.1 Objects

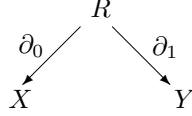
The objects of **Span(RGraph)** are (finite) *reflexive* graphs: that is, each object X consists of a set X_1 of *edges*, a set X_0 of *vertices*, and three functions, the *domain* function $d_0 : X_1 \rightarrow X_0$, (assigning the beginning vertex to an edge), the *codomain* function $d_1 : X_1 \rightarrow X_0$, (assigning the end vertex to an edge), and the *null* function $\epsilon : X_0 \rightarrow X_1$, (assigning a null loop to each vertex). Further there are axioms to be satisfied, namely that $d_0\epsilon = 1_{X_0}$ and $d_1\epsilon = 1_{X_0}$. There is an obvious notion of morphism of reflexive graphs. Since in this paper all graphs are reflexive we shall for brevity drop the adjective reflexive from now on. Graphs may be represented in the usual way as geometric figures with vertices and edges.

2.1.2 Operations on objects

There are two operations on objects which produce objects: (i) the products of graphs $X \times Y$ is the graph whose vertices (edges) are pairs of vertices (edges), one from X and one from Y ; the reverse graph X^{-1} of a graph X , which is actually just the graph X . It may seem strange to consider an operation whose effect is the identity, but it will enable us to make distinctions we need to make later on.

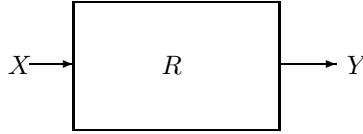
2.1.3 Arrows

Given graphs X and Y an arrow from X to Y consists of a graph R and two graph morphisms $\partial_0 : R \rightarrow X, \partial_1 : R \rightarrow Y$. Such an arrow is called a *span* of graphs. It is often denoted as follows:

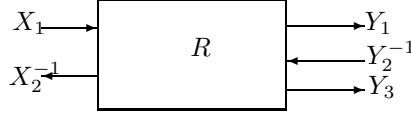


We call the graph R the head of the span and the morphisms ∂_0 and ∂_1 the two legs of the span. We will, by abuse of notation, denote the two legs of any span by the same symbols ∂_0 and ∂_1 . We will often denote the span simply as $R : X \rightarrow Y$. We call X and Y the domain and codomain, respectively, of R ; or the boundaries of R .

We represent a span $R : X \rightarrow Y$ by a picture of the following form:



We may also picture a span in another way. If the objects are given as products of graphs, for example $X = X_1 \times X_2^{-1}$, $Y = Y_1 \times Y_2^{-1} \times Y_3$ we picture the span as:



In this case we may call each of $X_1, X_2^{-1}, Y_1, Y_2^{-1}, Y_3$, boundaries of R .

2.1.4 Operations involving objects and arrows

2.1.5 Composition of spans.

The composite of spans $R : X \rightarrow Y$ and $S : Y \rightarrow Z$ is the span $R \bullet S : X \rightarrow Z$ whose head is the graph with vertex set

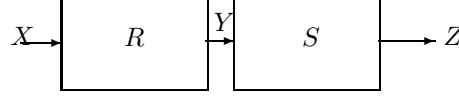
$$\{(r, s); r \text{ is a vertex of } R, s \text{ is a vertex of } S, \partial_1(r) = \partial_0(s)\}$$

and with edge set

$$\{(\rho, \sigma); \rho \text{ is a edge of } R, \sigma \text{ is a edge of } S, \partial_1(\rho) = \partial_0(\sigma)\}.$$

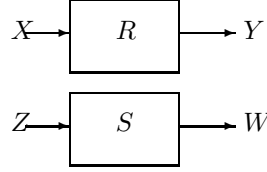
Beginnings and ends of edges have the obvious definitions. ($R \bullet S$ is the pullback $R \times_Y S$).

The pictorial representation of the composition of two spans R and S is as follows (with the obvious modification if the objects are products of graphs!):



Tensor of spans.

The tensor of two spans $R : X \rightarrow Y$ and $S : Z \rightarrow W$ is the span denoted $R \otimes S : X \times Z \rightarrow Y \times W$, and defined by: the head of $R \otimes S$ is $R \times S$; the legs of $R \otimes S$ are $\partial_0 \times \partial_0$ and $\partial_1 \times \partial_1$. The pictorial representation of the tensor of two spans is:



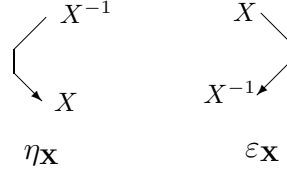
In addition to these operations there are the following constants of the algebra.

The spans $\eta_X : I \rightarrow X^{-1} \times X$ **and** $\varepsilon_X : X \times X^{-1} \rightarrow I$.

The terminal graph, denoted I , has one vertex, 0 and one edge, by necessity the null loop. The span with head X and legs $! : X \rightarrow I, \Delta : X \rightarrow X \times X$ is called η_X .

The span with head X and legs $\Delta : X \rightarrow X \times X, ! : X \rightarrow I$ is called ε_X .

The two spans are pictured thus:



The arrows η and ε are the unit and counit of the compact-closed structure on **Span(RGraph)**. It will become clear that their role in the context of this paper is to permit a feedback operation on distributed systems.

The correspondence between constants and operations, and the geometric representations given above, result in the fact that expressions in the algebra have corresponding circuit or system diagrams. We will draw such pictures later when discussing systems of accounts, though in this paper the pictures are solely an aid to reason and not precisely formalized. The formal thing is the expression.

2.1.6 2-cells of Span(RGraph)

If R and S are spans from X to Y , a 2-cell $\phi : R \rightarrow S$ consists of a graph morphism ϕ between the heads of the spans satisfying $\partial_0 \phi = \partial_0$, and $\partial_1 \phi = \partial_1$. Corresponding to each span R there is an obvious identity 2-cell $R \rightarrow R$. Further 2-cells compose in two ways, horizontally and vertically. Vertically, the composite $\phi \cdot \psi$ of $\phi : R \rightarrow S$ with $\psi : S \rightarrow T$ is formed by the composition of the graph

morphisms ϕ and ψ . Horizontally, the composite $\phi \bullet \psi$ of $\phi : R \rightarrow S : X \rightarrow Y$ with $\psi : T \rightarrow U : Y \rightarrow Z$ is formed by pullback.

2.2 Spans from graph morphisms

Given a graph morphism $f : X \rightarrow Y$ there are two special associated spans $f_* : X \rightarrow Y$ and $f^* : Y \rightarrow X$ defined as follows

$$\begin{aligned} f_* &= (1_X, f) : X \rightarrow Y, \\ f^* &= (f, 1_Y) : Y \rightarrow X. \end{aligned}$$

and two 2-cells $1_X \rightarrow f_* \bullet f^*$, $f^* \bullet f_* \rightarrow 1_Y$ exhibiting f^* as the right adjoint of f_* . We will need the following straightforward proposition in section 4.

Proposition

If $f : U \rightarrow X$ and $g : V \rightarrow Y$ are graph morphisms and $R : U \rightarrow V$, $S : X \rightarrow Y$ are spans of graphs then there is a bijection between 2-cell of spans

$$\phi : R \bullet g_* \rightarrow f_* \bullet S$$

and graph morphisms

$$\varphi : R \rightarrow S \text{ such that } \partial_0 \varphi = f_* \partial_0 \text{ and } \partial_1 \varphi = g_* \partial_1.$$

2.3 Behaviours of a span

Definition

A behaviour π of a span $R : X \rightarrow Y$ is a finite path in the graph R , the head of the span.

Notice that applying the legs of the span to a behaviour π yields two paths, one $\partial_0(\pi)$ in X and the other $\partial_1(\pi)$ in Y . The graphs X and Y may be thought of as the (left and right) boundaries of the system R . Then $\partial_0(\pi)$, $\partial_1(\pi)$ may be thought of as the behaviour of the boundaries of the system corresponding to the behaviour π , or the behaviour of the system reflected on the boundaries.

The following result is straightforward.

Proposition

(i) A behaviour of the composite $R; S$ of two spans is a pair of behaviours, one ρ of R , the other σ of S , such that $\partial_1(\rho) = \partial_0(\sigma)$. That is, a behaviour of $R; S$ consists of a behaviour of R and a behaviour of S which agree (synchronize) on the common boundary.

(ii) A behaviour of the tensor $R \otimes S$ of two spans is just a pair of behaviours, one ρ of R , the other σ of S .

(iii) A behaviour of $\eta_X : I \rightarrow X \times X$ is a path in X reflected (synchronously and equally) on the two boundaries. The behaviours of ε are similarly described.

3 Standard Accounts

We need first to define the notion of standard account (in which a general account will be valued).

Definition A channel is a graph with one vertex and edges being all non-negative integers. Suppose $X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_n$ are channels, and suppose that $X = X_1^{\xi_1} \times \dots \times X_m^{\xi_m}$ and $Y = Y_1^{\zeta_1} \times \dots \times Y_n^{\zeta_n}$ where the $\xi = (\xi_1, \dots, \xi_m)$ and $\zeta = (\zeta_1, \dots, \zeta_n)$ are sequences of 1's and -1's. Then the *standard account* $A_{X,Y}$ from X to Y (which is often denoted merely by A) is the span whose head vertex set is the integers, and an edge $\rho : r \rightarrow s$ is an $m+n$ -tuple of natural numbers $x_i (i = 1, 2, \dots, m), y_j (j = 1, 2, \dots, n)$ each satisfying

$$s - r = \sum_{i=1}^m \xi_i x_i - \sum_{j=1}^n \zeta_j y_j,$$

and $\partial_{0,i}(\rho) = x_i, \partial_{1,j}(\rho) = y_j$ where $\partial_{k,l}$ is ∂_k followed by the l 'th projection.

In words, a vertex of an account is a possible *value*, and an edge is a change of value of the account as a result of various ingoings and outgoings of value. The condition says that the change in the value of the account after a transaction is the result of the difference between ingoings and outgoings - that is, the condition is a *continuity equation* for value. Notice that value can only flow *into* the accounts on the channels of the form X^{+1} on the left, and on channels of the form X^{-1} on the right, and *out of* the accounts on channels of the form X^{-1} on the left, and on channels of the form X^{+1} on the right.

Note Standard accounts do not form a sub-algebra of **Span(RGraph)**; in particular, the collection of standard accounts does not contain identities and it is not closed under composition nor tensor. However, the class of standard accounts bears extra structure which will be used in the next section to define a compact closed bicategory of accounts. This structure is presented below in the form of data and axioms.

Before starting, note that if $\beta, \beta' : G \rightarrow A_{X,Y}$ are 2-cells from any span of graphs to an account such that β and β' take the same value on the vertices of G , then $\beta = \beta'$. In the following, we will define families of 2-cells in **Span(RGraph)** by only specifying their value on vertices. The reader can easily verify the existence of these 2-cells.

Data

1. For each account $A : X \rightarrow X$, let $\theta : 1_X \rightarrow A$ be the (unique) 2-cell which maps the only vertex of the head of 1_X to the vertex $0 \in A$.
2. For any pair of accounts $A_{X,Y}$ and $A_{Y,Z}$, let $\alpha : A_{X,Y} \bullet A_{Y,Z} \rightarrow A_{X,Z}$ be the 2-cell such that $\alpha(i, j) = i + j$.
3. For any pair of accounts $A_{W,X}$ and $A_{Y,Z}$, let $\tau : A_{W,X} \otimes A_{Y,Z} \rightarrow A_{W \otimes Y, X \otimes Z}$ be the 2-cell such that $\tau(i, j) = i + j$.

4. For each account $A : I \rightarrow X^{-1} \otimes X$, let $\delta : \eta_X \rightarrow A$ be the 2-cell which maps the only vertex of the head of η_X to the vertex $0 \in A$. (Of course, if $X = X_1^{\xi_1} \times \dots \times X_m^{\xi_m}$ then $X^{-1} = X_m^{-1 \times \xi_m} \times \dots \times X_1^{-1 \times \xi_1}$.)
5. For each account $A : X \otimes X^{-1} \rightarrow I$, let $\gamma : \epsilon_X \rightarrow A$ be the 2-cell which maps the only vertex of the head of ϵ_X to the vertex $0 \in A$.

Axioms

1. For any X and Y , the following 2-cells are equal.

$$(\theta \bullet A_{X,Y}) \cdot \alpha : (1_X \bullet A_{X,Y}) \rightarrow A_{X,X} \bullet A_{X,Y} \rightarrow A_{X,Y}$$

$$1_{A_{X,Y}} : A_{X,Y} \rightarrow A_{X,Y}$$

2. For any W, X, Y and Z , the following 2-cells are equal.

$$(\alpha \bullet A_{Y,Z}) \bullet \alpha : (A_{W,X} \bullet A_{X,Y}) \bullet A_{Y,Z} \rightarrow A_{W,Y} \bullet A_{Y,Z} \rightarrow A_{W,Z}$$

$$(A_{W,X} \bullet \alpha) \cdot \alpha : A_{W,X} \bullet (A_{X,Y} \bullet A_{Y,Z}) \rightarrow A_{W,X} \bullet A_{X,Z} \rightarrow A_{W,Z}$$

3. Given standard accounts $X \rightarrow Y$, $Y \rightarrow Z$, $X' \rightarrow Y'$ and $Y' \rightarrow Z'$, the following 2-cells are equal.

$$(\alpha \otimes \alpha) \cdot \tau : (A \bullet A) \otimes (A \bullet A) \rightarrow A \otimes A \rightarrow A$$

$$(\tau \bullet \tau) \cdot \alpha : (A \bullet A) \otimes (A \bullet A) \cong (A \otimes A) \bullet (A \otimes A) \rightarrow A \bullet A \rightarrow A$$

4. For any X , the following 2-cells are equal:

$$((\theta \otimes \delta) \bullet (\gamma \otimes \theta)) \cdot (\tau \bullet \tau) \cdot \alpha : (1_X \otimes \eta_X) \bullet (\epsilon_X \otimes 1_X) \rightarrow (A \otimes A) \bullet (A \otimes A) \rightarrow A \bullet A \rightarrow A$$

$$\theta : (1_X \otimes \eta_X) \bullet (\epsilon_X \otimes 1_X) = 1_X \rightarrow A$$

5. For any X , the following 2-cells are equal:

$$((\delta \otimes \theta) \bullet (\theta \otimes \gamma)) \cdot (\tau \bullet \tau) \cdot \alpha : (\eta_X \otimes 1_X) \bullet (1_X \otimes \epsilon_X) \rightarrow (A \otimes A) \bullet (A \otimes A) \rightarrow A \bullet A \rightarrow A$$

$$\theta : (\eta_X \otimes 1_X) \bullet (1_X \otimes \epsilon_X) = 1_{X^{-1}} \rightarrow A$$

Remark The structure described above is essentially that of a lax morphism of compact closed bicategories. (The domain of this morphism is the chaotic category whose objects are strings of plus and minus signs - the compact closed structure being the obvious one. The codomain is, of course, **Span(RGraph)**.) The first two data (and axioms) are part of the structure of a morphism of bicategories. The second half of the data (and axioms) relate to the compact closed structure. This abstract structure will be defined and investigated in detail in another paper.

4 The compact closed bicategory of accounts

In this section we make a formal definition of the compact-closed bicategory **Accounts** of *accounts*, in such a way that a *system of accounts with partita doppia* will be an expression in **Accounts**. The connection with conventional accounting will be made in the next section.

Definition

- An object of **Accounts** is a graph U together with a (reflexive) graph morphism $f : U \rightarrow X$ where X is a products of channels $X = X_1^{\xi_1} \times \dots \times X_m^{\xi_m}$.
- An arrow of **Accounts**, called a *general account*, from $f : U \rightarrow X$ to $g : V \rightarrow Y$ consists of a span $R : U \rightarrow V$, and a 2-cell $\phi_R : R \bullet g_* \rightarrow f_* \bullet A_{X,Y}$.

$$\begin{array}{ccc}
 U & \xrightarrow{R} & V \\
 f_* \downarrow & \phi_R \swarrow & \downarrow g_* \\
 X & \xrightarrow{A_{X,Y}} & Y
 \end{array}$$

- The composite of general accounts $(R : U \rightarrow V, \phi_R), (S : V \rightarrow W, \phi_S)$ is

$$(R \bullet S : U \rightarrow W, (R \bullet \phi_S) \cdot (\phi_R \bullet A) \cdot (f_* \bullet \alpha) : R \bullet S \bullet h_* \rightarrow f_* \bullet A).$$

Theorem The bicategory **Accounts** is compact closed.

proof The proof (details will be given elsewhere) amounts to first checking that left adjoint arrows in a compact closed bicategory **B** are the objects of a compact closed bicategory $\text{Ladj}(\mathbf{B})$, the arrows being squares in the bicategory containing a 2-cell. Combining this with the lax structure on standard accounts yields the required structure on **Accounts**.

Remark As a result of this theorem we may draw pictures of expressions of general accounts similar to those we described earlier for expressions in **Span(RGraph)**.

Definition A *system of accounts* is an expression in the compact closed bicategory **Accounts**. A *closed system of accounts* is an expression with domain and codomain of the form $!_U : U \rightarrow I$.

Corollary For a closed system of accounts there is an invariant of a behaviour, called the *total value*.

proof A closed system of accounts evaluates as a span $R : U \rightarrow V$ and a 2-cell $\phi_R : R \bullet (!_U)_* \rightarrow (!_V)_* \bullet A_{I,I}$. The 2-cell amounts to a graph morphism from R

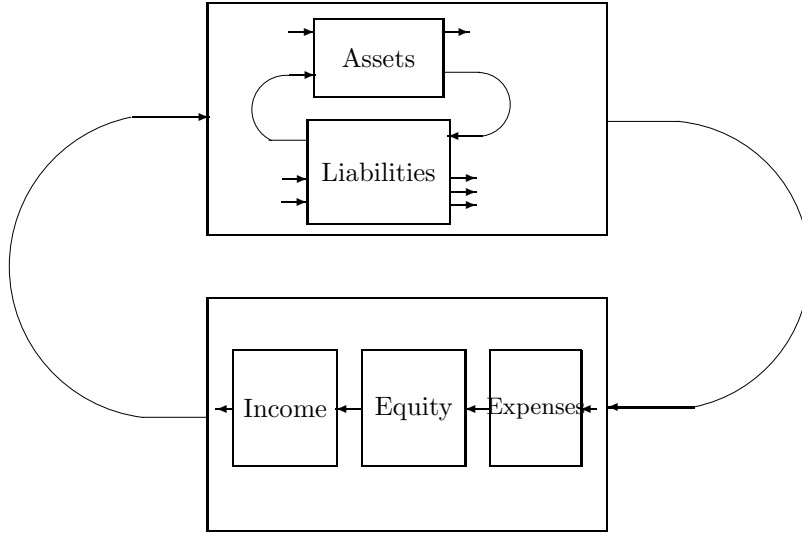
to $A_{I,I}$. But $A_{I,I}$ is a discrete reflexive graph, and so each path in R lies over a single vertex of $A_{I,I}$.

The fact that at this point the existence of an invariant is essentially trivial is a consequence (and an evidence for the naturality of) the structure of standard accounts.

5 The relation with conventional bookkeeping

A general account (R, ϕ_R) is that it is a transition system which is *measured* by a standard account, the measure being the 2-cell ϕ_R . The state of a conventional account may be much more than just its value; for example, an account may contain a record of its history. Further a transaction involving two accounts usually has much more information than just the value passed; for example, it may include the addresses of the people involved in the transaction.

A conventional accounting system is a closed system consisting of an expression in five different types of accounts - asset accounts, liability accounts, equity accounts, expense and income accounts, with the following schematic



An *asset* account is one in which the states have values in the non-negative integers, called debits; a *liability* account is one in which the states have values in the non-positive integers, called credits. Liability accounts generally record borrowings from external sources which need to be repaid. *Expense*, *income* and *equity* accounts exist in order to produce a closed system. Expense and income accounts record outgoings and ingoings from external sources which are free of any further obligation. Expense accounts transactions reduce assets and increase expenses, and hence expense accounts have debit state; income account transactions increase assets and reduce income and hence income accounts have credit states. In a transaction involving the ingoing of income an asset account

becomes more positive while the income account becomes correspondingly more negative. The composite of the three (types of) accounts, income, expense and equity may be called owner's equity and represents (with negative sign) the total value of the business. At least once a year the income and expense accounts are zeroized - that is, value is transferred from the expense accounts to the equity accounts, and from the equity accounts to the income accounts, thereby placing all the owner's equity in the equity accounts.

The behaviours considered have invariant (total) 0. When the expense and income accounts have been zeroized (or composed with the equity account) this means the invariant is expressed by the conventional equation

$$Assets = Liabilities + Owner's Equity$$

In this equation Liabilities and Owner's Equity the negative of the totals in the liability and equity accounts, respectively. The Balance Sheet contains the values of the states of the asset, liability, and owner's equity accounts.

Notice that it is crucial in defining a system of accounts that the diagonal operation (a useful operation in describing other concurrent systems) must be avoided. Copying money is counterfeiting and leads to a failure of the invariant.

An Example

Consider a system of accounts with one account of each type discussed. Suppose initially the assets are 1000, and hence the equity is -1000. Then consider the following sequence of transactions - first a purchase of 2000 value of consumable goods is made incurring a liability of -2000. Next there is income of 1500, as a result of which the asset account increases by 1500, and the income account reduces by the same amount. Next 1000 of assets are used to reduce the liability. Next the expenses are zeroized, and then the same for the income. At the end of this sequence of transactions there are 1500 debit of assets, 1000 credit of liabilities, and an owner's equity of 500 credit.

Asset, Liability, Expense, Income, Equity

(1000, 0, 0, 0, -1000)

↓

(1000, -2000, 2000, 0, -1000)

↓

(2500, -2000, 2000, -1500, -1000)

↓

(1500, -1000, 2000, -1500, -1000)

↓

(1500, -1000, 0, -1500, 1000)

↓

(1500, -1000, 0, 0, -500)

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